

Bohmian Mechanics Revisited

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Abstract: We consider the problem of whether there are deterministic theories describing the evolution of an individual physical system in terms of the definite trajectories of its constituent particles and which stay in the same relation to Quantum Mechanics as Bohmian Mechanics but which differ from the latter for what concerns the trajectories followed by the particles. Obviously, one has to impose on the hypothetical alternative theory precise physical requirements. We analyse various such constraints and we show step by step how to meet them. This way of attacking the problem turns out to be useful also from a pedagogical point of view since it allows to recall and focus on some relevant features of Bohm's theory. One of the central requirements we impose on the models we are going to analyse has to do with their transformation properties under the transformations of the extended Galilei group. In a context like the one we are interested in one can put forward various requests that we refer to as physical and genuine covariance and invariance. Other fundamental requests are that the theory allows the description of isolated physical systems as well as that it leads to a solution (in the same sense as Bohmian Mechanics) of the measurement problem. We show that, even when all above conditions are taken into account, there are infinitely many inequivalent (from the point of view of the trajectories) bohmian-like theories reproducing the predictions of Quantum Mechanics. This raises some interesting questions about the meaning of Bohmian Mechanics.

1 Introduction

The reasons which have led David Bohm to the formulation of his deterministic hidden variable theory have been expressed with great lucidity by John Bell in *Against Measurement* [1]:

“In the beginning, Schrödinger tried to interpret his wavefunction as giving somehow the density of stuff of which the world is made. He tried to think of an electron as represented by a wavepacket ... a wavefunction appreciably different from zero only over a small region in space. The extension of that region he thought of as the actual size of the electron ... his electron was a little bit fuzzy. At first he thought that small wavepackets, evolving according to the Schrödinger equation, would remain small. But that was wrong. Wavepackets diffuse, and with the passage of time become indefinitely extended, according to the Schrödinger equation. But however far the wavefunction has extended, the reaction of a detector to an electron remains spotty. So Schrödinger’s realistic interpretation did not survive.

Then came the Born interpretation. The wavefunction gives not the density of stuff, but gives rather (on squaring its modulus) the density of probability. Probability of what, exactly? Not of the electron being there, but of the electron being found there, if its position is “measured”.

Why this aversion to “being” and insistence on “finding”? The founding fathers were unable to form a clear picture of things on the remote atomic scale. They became very aware of the intervening apparatus, and of the need for a “classical” base from which to intervene on the quantum system. And so the shifty split.

The kinematics of the world, in this orthodox picture is given by a wavefunction (maybe more than one?) for the quantum part, and classical variables ... variables which have values ... for the classical part:

$$(\Psi(t, q, \dots), X(t), \dots)$$

The X’s are somehow macroscopic. This is not spelled out very explicitly. The dynamics is not very precisely formulated either. It includes a Schrödinger equation for the quantum part, and some sort of classical mechanics for the classical part, and “collapse” recipes for their interactions.

It seems to me that the only hope of precision with this dual (Ψ, x) kinematics is to omit completely the shifty split, and let both Ψ and $x(t)$ refer to the world as a whole. Then the x ’s must not be confined to some vague macroscopic scale, but must extend to all scales. In the picture of de Broglie and Bohm, every particle is attributed a position $x(t)$. Then instrument pointers ... assemblies of particles, have positions, and experiments have results. The dynamics is given by the world Schrödinger equation *plus* precise “guiding” equations prescribing how the $x(t)$ ’s move under the influence of Ψ .

These sentences describe the essence of Bohm's theory in a very lucid way. We consider it unnecessary to comment on its relevance and on the crucial role it has played for the debate on the conceptual foundations of Quantum Mechanics. As everybody knows, it allowed a deeper understanding of some basic features of the quantum aspects of natural phenomena. First of all, it has put into evidence the unappropriateness of von Neumann's conclusion about the impossibility of a deterministic completion of Quantum Mechanics. Secondly, it has been the main stimulus for Bell's investigations which have compelled the scientific community to face the nonlocal aspects of nature.

It is the purpose of this paper to investigate whether the bohmian program of reproducing the predictions of Quantum Mechanics within a framework in which particles have definite trajectories leads unavoidably, when some necessary general requirements are taken into account, to Bohmian Mechanics, or if other theories exhibiting the same features can be devised. The conclusion will be that there exists actually infinitely many such theories. This poses some interesting problems for the bohmian theory itself.

2 A short survey of Bohmian Mechanics and its more relevant features

Bohmian theory [2,3,4] is a hidden variable theory describing the states of individual physical systems in terms of their wavefunction $\Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t)$ in configuration space and the actual positions $\mathbf{Q}_k(t)$ of all particles of the system. The rules of the game are quite simple:

- Assign $\Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; 0)$ and $\mathbf{Q}_k(0)$.
- Consider the two evolution equations:

$$i\hbar \frac{\partial \Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t)}{\partial t} = \hat{H} \Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t) \quad (2.1)$$

$$\frac{d\mathbf{Q}_k(t)}{dt} = \frac{\hbar}{m_k} \Im \frac{\Psi^*(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t) \nabla_k \Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t)}{|\Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t)|^2} \Big|_{\mathbf{q}_i=\mathbf{Q}_i} \quad (2.2)$$

- Solve (2.1) for the initial condition $\Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; 0)$, insert the solution in the r.h.s. of (2.2) and solve it for the initial conditions $\mathbf{Q}_k(0)$. In this way one uniquely determines $\mathbf{Q}_k(t)$.

The fundamental feature of the theory consists in the fact that if consideration is given to an ensemble of physical systems (all described by the same wavefunction $\Psi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N; t)$) each containing N particles whose positions are¹ $\mathbf{Q} = (\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_N)$ (evolving as prescribed by Eq. (2.2)) and such that the probability density $\rho(\mathbf{Q}, t)$ for the configuration \mathbf{Q} satisfies:

$$\rho(\mathbf{Q}, 0) = |\Psi(\mathbf{Q}, 0)|^2 \quad (2.3)$$

then, the constituent particles of the systems of the ensemble follow trajectories such that one has, at all times:

$$\rho(\mathbf{Q}, t) = |\Psi(\mathbf{Q}, t)|^2. \quad (2.4)$$

The fact that Eq. (2.3) implies Eq. (2.4) is referred to as *equivariance* of the functional $\rho^\Psi = |\Psi|^2$. One can easily prove it as follows. If one considers the dynamical law specified by the velocity field² $\mathbf{v}_{Bk}(\mathbf{Q}; t)$:

$$\frac{d\mathbf{Q}_k(t)}{dt} = \mathbf{v}_{Bk}(\mathbf{Q}; t); \quad (2.5)$$

then, as is well known, the probability density $\rho(\mathbf{Q}; t)$ for the configuration \mathbf{Q} obeys the continuity equation

$$\frac{\partial \rho(\mathbf{Q}; t)}{\partial t} + \sum_{k=1}^N \nabla_k \cdot [\rho(\mathbf{Q}; t) \mathbf{v}_{Bk}(\mathbf{Q}; t)] = 0. \quad (2.6)$$

There follows that if we choose

$$\mathbf{v}_{Bk}(\mathbf{Q}; t) = \frac{\mathbf{j}_{QMk}(\mathbf{Q}; t)}{|\Psi(\mathbf{Q}; t)|^2}, \quad (2.7)$$

where $\mathbf{j}_{QMk}(\mathbf{Q}; t)$ is the quantum mechanical current density for particle k :

$$\mathbf{j}_{QMk}(\mathbf{Q}; t) = \frac{\hbar}{m_k} \Im[\Psi^*(\mathbf{Q}; t) \nabla_k \Psi(\mathbf{Q}; t)], \quad (2.8)$$

¹From now on we will similarly use the shorthand \mathbf{q} for $(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$.

²Eq. (2.5) is meaningful only if the velocity field is assumed to be one-valued and to satisfy appropriate regularity properties which we will discuss below.

then the solution of equation (2.6) (given the initial condition (2.3)) is $\rho(\mathbf{Q}, t) = |\Psi(\mathbf{Q}, t)|^2$ as it is easily seen by taking into account the quantum continuity equation:

$$\frac{\partial |\Psi(\mathbf{Q}; t)|^2}{\partial t} + \sum_{k=1}^N \nabla_k \cdot [\mathbf{j}_{QMk}(\mathbf{Q}; t)] = 0. \quad (2.9)$$

Before concluding this brief account of Bohmian Mechanics we would like to recall some relevant points:

- In the bohmian spirit, Standard Quantum Mechanics (SQM) is considered as an incomplete theory. To describe an individual system, besides its wavefunction one needs further (hidden) variables: the positions! The repetition of the preparation procedure leading to $\Psi(\mathbf{q}; 0)$ leads to a distribution of the hidden variables such that (2.3) holds. The precise trajectories followed by the particles are such to reproduce at all times the distribution of the outcomes of position measurements predicted by SQM.
- Each particle of the universe has at all times a precise position.
- No other assumption is made: there are no measurements, no wavepacket reductions and so on.
- There are contextual aspects giving rise to problems of remarkable epistemological relevance if one pretends that knowledge of the hidden variables determines the precise value of any conceivable observable of the standard formulation. All such problems are overcome simply by taking the attitude that what the theory is about are the positions and the trajectories of all particles and nothing else. With reference to the remarks by Bell in the above quotation, the theory claims that $|\Psi(\mathbf{Q}; t)|^2$ gives the probability density of the configuration *being* \mathbf{Q} at time t and not, as SQM, the probability density of *finding* the outcome \mathbf{Q} *if a measurement is performed*.

We would also like to point out that, recently, there have been extremely interesting investigations [5] aimed to clarify the so called Quantum Equilibrium Hypothesis, i.e. in which sense the initial condition $\rho(\mathbf{Q}, 0) = |\Psi(\mathbf{Q}, 0)|^2$ comes about. Note that if this condition is not satisfied by the members of the

ensemble and bohmian evolution equations hold, then the actual distribution at time t would contradict the predictions of SQM. Similarly, there have been detailed analysis [6] proving the uniqueness and the global existence of the solutions of the dynamical equation (2.5) for the trajectories, a highly non-trivial problem due, e.g., to the possible vanishing of the wavefunction or to its behaviour at infinity. It is important to stress that such investigations refer to the theory as it has been presented, i.e. with the prescription that the velocity field is precisely the one defined above in terms of the wavefunction.

This last remark has to be kept in mind to fully appreciate the problem to which the present paper is addressed: we will not question the uniqueness of the solutions for a given velocity field, but of the velocity field itself, i.e. we will face the problem of *whether there are theories which are equivalent to SQM in the same sense of Bohmian Mechanics, but which describe the evolution in terms of different trajectories.*

3 Is Bohmian Mechanics Unique?

As just mentioned, the question we are interested in is the following. Suppose we devise a theory which completes Quantum Mechanics by exhibiting the feature that all particles follow definite trajectories and thus have at any time definite positions. The only allowed ingredients of the theory are the wavefunction and the particle positions. Does such a request lead unavoidably to Bohmian Mechanics? That this is not the case can be shown in a very elementary way. One keeps the statevector and the Schrödinger evolution equation for it and one changes the dynamical equation for the trajectories by adding to the bohmian velocity term $\mathbf{v}_{Bk}(\mathbf{Q}; t)$ an additional velocity field $\mathbf{v}_{Ak}(\mathbf{Q}; t)$ given by:

$$\mathbf{v}_{Ak}(\mathbf{Q}; t) = \frac{\mathbf{j}_{Ak}(\mathbf{Q}; t)}{|\Psi(\mathbf{Q}; t)|^2}, \quad (3.1)$$

where $\mathbf{j}_{Ak}(\mathbf{Q}; t)$ satisfies:

$$\sum_{k=1}^N \nabla_k \cdot \mathbf{j}_{Ak}(\mathbf{Q}; t) = 0 \quad (3.2)$$

One can then argue as follows. The equation:

$$\frac{d\mathbf{Q}_k(t)}{dt} = \mathbf{v}_{Bk}(\mathbf{Q}; t) + \mathbf{v}_{Ak}(\mathbf{Q}; t) \quad (3.3)$$

implies

$$\frac{\partial \rho(\mathbf{Q}; t)}{\partial t} + \sum_{k=1}^N \nabla_k \cdot \{\rho(\mathbf{Q}; t)[\mathbf{v}_{Bk}(\mathbf{Q}; t) + \mathbf{v}_{Ak}(\mathbf{Q}; t)]\} = 0. \quad (3.4)$$

It is then obvious that the function

$$\rho(\mathbf{Q}, t) = |\Psi(\mathbf{Q}; t)|^2 \quad (3.5)$$

is the solution of Eq. (3.4) satisfying the condition $\rho(\mathbf{Q}, 0) = |\Psi(\mathbf{Q}, 0)|^2$ as a consequence of the SQM continuity equation since, with the choice (3.5), the last term of Eq. (3.4) vanishes in virtue of (3.2). We stress that Eq. (3.3) implies that, in general, the trajectories are different from those of Bohmian Mechanics which are characterised by $\mathbf{v}_{Ak}(\mathbf{Q}; t) = 0$.

Obviously, one must be particularly careful in choosing the new velocity field to guarantee the existence, uniqueness and globality of the solution. Even more, the regularity conditions for the velocity field which make legitimate to pass from Eq. (3.3) to (3.4) could fail to hold. These problems will be dealt with in the next Sections.

We mention that E. Squires [7] has considered a modification of this type in which $\mathbf{j}_{Ak}(\mathbf{Q}; t)$ was simply a constant current. Unfortunately, such a choice implies that there are particles which flow in and out at infinity, so that particles can be “created” and/or “destroyed”, a physically unacceptable fact in the present context. Moreover, the theory turns out to be non-covariant (see below). For these reasons Squires himself has considered unviable the line of thought we are pursuing here.

4 Sufficient conditions for the velocity field

The considerations of the final part of the previous Section could lead one to think that the necessary mathematical and physical requirements which have to be imposed to the theory, would reduce the family of possible dynamical models we are envisaging to Bohmian Mechanics itself. As we shall show, this is not the case, but, on the contrary, there are infinitely many sensible bohmian-like model theories with different trajectories.

For what concerns the mathematical properties of the velocity field which guarantee the existence of the trajectories and the validity of the continuity

equation, we do not want to be too technical. We will limit ourselves to point out that if the additional velocity field is defined everywhere and continuous and not diverging more than linearly at infinity, no particle can reach infinity in a finite time, the flux vanishes at infinity and thus the model is physically sensible. In refs. [6] the implications of the occurrence of possible singularities of the velocity field in the standard bohmian framework have been discussed and proved to have no unacceptable consequences. Here we will not go through such a detailed mathematical analysis of the properties of the additional velocity fields and we will suppose that the expressions we will introduce do not give rise to any problem.

In the next Section we start by imposing necessary physical requests on the models we are envisaging by analysing the delicate problem, within a context like ours, of the appropriate covariance and invariance requirements for the theory.

5 Covariance and Invariance: Genuine versus Physical

It is important to focus on the particular aspects that the problem of covariance acquires in a context like Bohmian Mechanics. We will obviously deal with the transformations of the Galilei group. There are different requests that one can put forward and that we will denote as genuine and physical covariance requirements. In fact, one should keep clearly in mind, as shown in ref. [5], that as a consequence of absolute uncertainty, the hidden variables of the theory are inaccessible. Thus, if one considers as meaningful only what the theory claims to be physically accessible, one could be satisfied with the request that different observers agree on the “objective” probability density for the various configurations at a given time. Since the class of theories we are envisaging are assumed to give configuration probability densities in agreement with those of Standard Quantum Mechanics, such “physical” covariance conditions turn out to be automatically satisfied.

However, as Bell [8] has appropriately stressed, for a theory like the one under consideration one should impose requests which go beyond the one of simple agreement with experiment. He demands that the theory does not admit any (even hidden) preferred reference frame. In simpler words, what

Bell requires is that observers related to one another by a transformation of the Galilei group agree on the (unaccessible) trajectories that the particles follow. If this happens the theory is said to be genuinely covariant.

In the case in which the underlying quantum problem exhibits invariance under the transformations of the Galilei group, a similar distinction could be made with reference to the bohmian-like models we are considering. In particular, in such a case, the request of genuine invariance amounts to demanding that observers connected by a transformation of the Galilei group which consider systems in the “same” — in their own language — initial conditions (thus corresponding to objectively different physical situations) will agree on the trajectory that each individual particle follows.

In our case, since the Schrödinger equation already satisfies the covariance and/or invariance requests, we can limit our considerations to the properties of the velocity fields determining the trajectories. We will discuss in detail only the case of special Galilei transformations and we will mention the transformation properties of the velocity field for the other transformations of the Galilei group. As we will see, Bohmian Mechanics in its standard form is physically and genuinely covariant and, consequently, invariant if the corresponding quantum problem is invariant under the transformations of the Galilei group.

6 Special Galilei Transformations

Let us discuss, first of all, how the bohmian velocity field $\mathbf{v}_{Bk}(\mathbf{Q}; t)$ changes under a special Galilei transformation:

$$\begin{cases} \mathbf{q}'_i = \mathbf{q}_i - \mathbf{v}t \\ t' = t. \end{cases} \quad (6.1)$$

To this purpose let us suppose that an observer O considers a specific initial wavefunction $\Psi(\mathbf{q}, 0)$ and a specific configuration $\mathbf{Q}(0)$. By using Eq. (2.1) and the precise prescription (2.7) of the theory, he determines the wavefunction $\Psi(\mathbf{q}, t)$ and the velocity field $\mathbf{v}_{Bk}(\mathbf{Q}; t)$. In turn, such a field, when inserted into (2.5), uniquely determines the trajectory $\mathbf{Q}(t)$. Another observer O' considers the same *objective* wavefunction³ in his language $\Psi'(\mathbf{q}'; 0) = \Psi(\mathbf{q}; 0)$ and uses the same formal procedure as the original observer (i.e. it leaves $\Psi'(\mathbf{q}'; 0)$ evolve

³Obviously, in the case of an arbitrary transformation of the group in place of Eq. (6.1), the functional dependence of Ψ' from its argument will be different from the one of Ψ . Note,

according to *his* Hamiltonian, which may be different from the one of the original observer), evaluates the corresponding velocity field $\mathbf{v}'_{Bk}(\mathbf{Q}'; t')$ and studies the evolution of the particle which initially is at the same *objective* point.

We recall [9] that the Galilei transformation (6.1) is implemented by the unitary operator

$$\mathcal{G}_{\mathbf{v}}(t) = e^{-\frac{i}{2\hbar} M v^2 t} e^{-\frac{i}{\hbar} M \hat{\mathbf{Q}} \cdot \mathbf{v}} e^{\frac{i}{\hbar} \hat{\mathbf{P}} \cdot \mathbf{v} t}, \quad (6.2)$$

$M = \sum_{i=1}^N m_i$ being the total mass and $\hat{\mathbf{Q}}, \hat{\mathbf{P}}$ the centre-of-mass position and momentum operators, respectively.

Equation (6.2) shows that the wavefunctions for O', in his variables \mathbf{q}'_i and t' , is:

$$\Psi'(\mathbf{q}'; t') = \langle \mathbf{q}' | \mathcal{G}_{\mathbf{v}}(t') | \Psi, t \rangle = e^{-\frac{i}{2\hbar} M v^2 t'} e^{-\frac{i}{\hbar} M \mathbf{q}' \cdot \mathbf{v}} \Psi(\mathbf{q}'_i + \mathbf{v} t', t'). \quad (6.3)$$

As we have seen in Section 2, the observer O determines the particle trajectories according to:

$$\frac{d\mathbf{Q}_k(t)}{dt} = \mathbf{v}_{Bk}(\mathbf{Q}; t), \quad \mathbf{v}_{Bk}(\mathbf{Q}; t) = \frac{\hbar}{m_k} \Im \frac{\Psi^*(\mathbf{q}; t) \nabla_k \Psi(\mathbf{q}; t)}{|\Psi(\mathbf{q}; t)|^2} \Big|_{\mathbf{q}=\mathbf{Q}} \quad (6.4)$$

and the observer O' determines his own velocity field $\mathbf{v}'_{Bk}(\mathbf{Q}'; t')$ by using the corresponding prescription, getting:

$$\begin{aligned} \mathbf{v}'_{Bk}(\mathbf{Q}'; t') &= \frac{\hbar}{m_k} \Im \frac{\Psi'^*(\mathbf{q}'; t') \nabla'_k \Psi'(\mathbf{q}'; t')}{|\Psi'(\mathbf{q}'; t')|^2} \Big|_{\mathbf{q}'=\mathbf{Q}'} \\ &= \frac{\hbar}{m_k} \Im \frac{\Psi(\mathbf{q}'_i + \mathbf{v} t', t') \nabla'_k \Psi(\mathbf{q}'_i + \mathbf{v} t', t')}{|\Psi(\mathbf{q}'; t')|^2} \Big|_{\mathbf{q}'=\mathbf{Q}'} + \frac{\hbar}{m_k} \Im \left(-\frac{i}{\hbar} m_k \mathbf{v} \right) \\ &= \mathbf{v}_{Bk}(\mathbf{Q}' + \mathbf{v} t'; t') - \mathbf{v}. \end{aligned} \quad (6.5)$$

Since O' uses the equation:

$$\frac{d\mathbf{Q}'_k(t')}{dt'} = \mathbf{v}'_{Bk}(\mathbf{Q}'; t') = \mathbf{v}_{Bk}(\mathbf{Q}' + \mathbf{v} t'; t') - \mathbf{v}, \quad (6.6)$$

if $\mathbf{Q}_k(t)$ satisfies the first of Eqs. (6.4), then

$$\mathbf{Q}'_k(t') = \mathbf{Q}_k(t) - \mathbf{v} t \quad (6.7)$$

in fact, that the specific relation between Ψ and Ψ' derives also from the fact that, according to Eq. (6.1), for $t = 0$ $\mathbf{q}' = \mathbf{q}$.

satisfies (6.6).

This proves the genuine covariance of the trajectories. Obviously, when the underlying quantum theory is invariant for the transformation considered, the Hamiltonian for O' coincides with the one for O and the above procedure shows that if one considers the same⁴ wavefunction and the same initial conditions in the two reference frames, one gets exactly the same trajectories.

The conclusion is obvious: Bohmian Mechanics exhibits genuine covariance and/or invariance for special Galilei transformations. We simply list here the transformation properties which characterise the bohmian velocity field for all the transformations of the extended Galilei group:

$$\begin{aligned}
\text{i. Space translations: } & \mathbf{v}'_{Bk}(\mathbf{Q}'; t') = \mathbf{v}_{Bk}(\mathbf{Q}' + \mathbf{A}; t') \\
\text{ii. Time translations: } & \mathbf{v}'_{Bk}(\mathbf{Q}'; t') = \mathbf{v}_{Bk}(\mathbf{Q}'; t' + \tau) \\
\text{iii. Space rotations: } & \mathbf{v}'_{Bk}(\mathbf{Q}'; t') = \mathcal{R}\mathbf{v}_{Bk}(\mathcal{R}^{-1}\mathbf{Q}'; t') \\
\text{iv. Galilei boosts: } & \mathbf{v}_{Bk}(\mathbf{Q}'; t') = \mathbf{v}_{Bk}(\mathbf{Q}' + \mathbf{v}t'; t') - \mathbf{v} \\
\text{v. Space reflection: } & \mathbf{v}'_{Bk}(\mathbf{Q}'; t') = -\mathbf{v}_{Bk}(-\mathbf{Q}'; t') \\
\text{vi. Time reversal: } & \mathbf{v}'_{Bk}(\mathbf{Q}'; t') = -\mathbf{v}_{Bk}(\mathbf{Q}'; -t').
\end{aligned} \tag{6.8}$$

These, in turn, guarantee the genuine covariance and/or invariance of the theory for the extended Galilei group.

7 Looking for alternative models

As already stated, it is possible to find a whole family of model theories which attribute definite trajectories to the particles and, with respect to SQM, stay in the same relation as Bohm's theory. With reference to the case of a system of N particles and to the formalism of Section 3, let us suppose that we choose for the additional velocity field $\mathbf{v}_{Ak}(\mathbf{Q}; t)$ the following expression:

$$\begin{aligned}
\mathbf{v}_{Ak}(\mathbf{Q}; t) &= \gamma_k \frac{\nabla_k \times \{|\Psi(\mathbf{q}; t)|^2 [\mathbf{v}_{Bk}(\mathbf{q}; t)]\}}{|\Psi(\mathbf{q}; t)|^2} \Big|_{\mathbf{q}=\mathbf{Q}} \\
&= \frac{i\hbar\gamma_k}{m_k} \frac{\nabla_k \Psi}{\Psi} \times \frac{\nabla_k \Psi^*}{\Psi^*} \Big|_{\mathbf{q}=\mathbf{Q}}.
\end{aligned} \tag{7.1}$$

⁴Here, by the expression “the same”, we mean that the functional dependence of the initial wavefunction and the numerical value of the initial position is the same for the two observers. Note that this means that they are considering two “objectively different” physical situations.

In Eq. (7.1) γ_k is an appropriate dimensional constant (we can choose it to be $[\hbar/m_k c]$ without introducing any new constant of nature in the theory) and $\nabla_k \times$ denotes the curl differential operator over the coordinates of the k -th particle. The velocity field (7.1) is easily proved to exhibit the same transformation properties (6.8i)–(6.8iii) of the bohmian velocity field. On the contrary, its transformation properties under Galilei boosts are such to violate the covariance requirements for the theory. To take seriously the model we are proposing we have then, first of all, to overcome such a difficulty. As a first attempt to reach this goal, let us start by considering a system of N spinless particles interacting via a potential which depends only on the modulus of the difference of their coordinates so that one can be tempted to adopt the following line of thought. As is well known, in such a case the bohmian dynamics is equivalent to the one obtained by resorting to the centre-of-mass (\mathbf{R}) and relative (\mathbf{r}_k) coordinates:

$$i\hbar \frac{\partial \Psi(\mathbf{R}, \mathbf{r}_k; t)}{\partial t} = \hat{H} \Psi(\mathbf{R}, \mathbf{r}_k; t), \quad (7.2)$$

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2M} + \sum_k \frac{\hat{\mathbf{p}}_k^2}{2\mu_k} + V, \quad (7.3)$$

$$\frac{d\tilde{\mathbf{R}}(t)}{dt} = \frac{\hbar}{M} \Im \frac{\Psi^*(\mathbf{R}, \mathbf{r}_k; t) \nabla_R \Psi(\mathbf{R}, \mathbf{r}_k; t)}{|\Psi(\mathbf{R}, \mathbf{r}_k; t)|^2} \Big|_{\mathbf{R}=\tilde{\mathbf{R}}, \mathbf{r}_k=\tilde{\mathbf{r}}_k} = \mathbf{v}_{BR}(\tilde{\mathbf{R}}, \tilde{\mathbf{r}}_k; t), \quad (7.4)$$

$$\frac{d\tilde{\mathbf{r}}_k(t)}{dt} = \frac{\hbar}{\mu_k} \Im \frac{\Psi^*(\mathbf{R}, \mathbf{r}_k; t) \nabla_{r_k} \Psi(\mathbf{R}, \mathbf{r}_k; t)}{|\Psi(\mathbf{R}, \mathbf{r}_k; t)|^2} \Big|_{\mathbf{R}=\tilde{\mathbf{R}}, \mathbf{r}_k=\tilde{\mathbf{r}}_k} = \mathbf{v}_{Bk}(\tilde{\mathbf{R}}, \tilde{\mathbf{r}}_k; t), \quad (7.5)$$

where we have denoted by M and μ_k the total and reduced masses respectively.

This fact suggests to overcome the difficulty by keeping the first three of the above equations and changing the remaining ones by adding to the bohmian velocity fields of the relative motions new terms analogous to the one of Eq. (7.1), so that Eqs. (7.5) are replaced by:

$$\frac{d\tilde{\mathbf{r}}_k(t)}{dt} = \mathbf{v}_{Bk}(\tilde{\mathbf{R}}, \tilde{\mathbf{r}}_k; t) + \mathbf{v}_{Ak}(\tilde{\mathbf{R}}, \tilde{\mathbf{r}}_k; t) \equiv \mathbf{v}_{Nk}(\tilde{\mathbf{R}}, \tilde{\mathbf{r}}_k; t), \quad (7.6)$$

where

$$\mathbf{v}_{Ak}(\tilde{\mathbf{R}}, \tilde{\mathbf{r}}_k; t) := \gamma_k \frac{\nabla_{r_k} \times [|\Psi(\mathbf{R}, \mathbf{r}_k; t)|^2 \mathbf{v}_{Bk}(\mathbf{R}, \mathbf{r}_k; t)]}{|\Psi(\mathbf{R}, \mathbf{r}_k; t)|^2} \Big|_{\mathbf{R}=\tilde{\mathbf{R}}, \mathbf{r}_k=\tilde{\mathbf{r}}_k}. \quad (7.7)$$

The velocity fields of the just considered model obviously have the correct behaviour under the transformations of the proper Galilei group and, in virtue of Eq. (7.6) the trajectories of the individual particles differ from those of Bohmian Mechanics.

Up to now we have not considered the transformations of the extended group. It is easily seen that the velocity fields of Eq. (7.7) transform like pseudovectors under space reflection so that the theory is not covariant for the full group. One could then introduce quantities like

$$f_k^\Psi(t) := i \frac{d}{dt} \int d^{3(N-1)} r d^3 R \left\{ [\nabla_k \Psi(\mathbf{R}, \mathbf{r}_k; t) \times \nabla_k \Psi^*(\mathbf{R}, \mathbf{r}_k; t)] \cdot \frac{\mathbf{r}_k}{r_k} \right\}. \quad (7.8)$$

Such functions are continuous, real scalar functions of t , which change sign for space reflections and do not change sign for time reversal. Therefore, if we define new velocity fields:

$$\tilde{\mathbf{v}}_{Ak}(\mathbf{R}, \mathbf{r}_k; t) = f_k^\Psi(t) \mathbf{v}_{Ak}(\mathbf{R}, \mathbf{r}_k; t) \quad (7.9)$$

and we replace in all previous formulae $\mathbf{v}_{Ak}(\mathbf{R}, \mathbf{r}_k; t)$ with $\tilde{\mathbf{v}}_{Ak}(\mathbf{R}, \mathbf{r}_k; t)$, we have the appropriate transformation properties of the additional velocities under the transformations of the full Galilei group.

8 Some serious drawbacks of the model of Section 7

The model theory we have just presented seems, at first sight, physically sensible and satisfies all covariance requirements one can put forward for it. Obviously, the model exhibits a formal feature which could be considered as not fully satisfactory, i.e. its physical implications (more precisely the precise trajectories followed by the various particles) depend on the particular choice one makes for the relative coordinates. Said differently, if one denotes by A a precise transformation leading from the absolute $\{\mathbf{q}_i\}$ to the relative $\{\mathbf{R}, \mathbf{r}_i\}$ coordinates, and one resorts to Eqs. (7.2–7.4) and (7.6–7.7) to define the dynamics, the evolution of another set of relative variables $\{\mathbf{R}, \hat{\mathbf{r}}_i\}$ related to $\{\mathbf{q}_i\}$ by a different transformation \hat{A} is governed by velocity fields which have not the formal structure (7.7). This, by itself, does not represent a drawback

of the model⁵; rather one could see it as a further direct and simple proof that one can exhibit many different theories of the kind we are looking for.

However, if one analyses in a more critical way the proposed model, one easily realises that its characteristic dependence on the choice made for the relative variables, beside making it formally “less elegant” than Bohmian Mechanics, has some unacceptable physical implications. Suppose in fact that the particles of the physical system under consideration can be arranged in various groups, such that the members of different groups are non-interacting and non-entangled (this last specification referring to the statevector of the system in the absolute coordinates). In such a case SQM as well as Bohmian Mechanics allow one to consider the physical systems composed by the particles of any group as isolated from the others (at least until the evolution brings particles of different groups into interaction). Accordingly, the evolution of the particles of one group does not depend in any way whatsoever on the members of other groups. If this does not happen, one could legitimately claim that the model exhibits “more” nonlocal aspects than those which are unavoidably brought into play by quantum entanglement.

It should be evident that the model of the previous Section does actually violate the requirement we have just put forward. In fact, if one writes it in the absolute coordinates, one sees that the additional velocity corresponding to the variable \mathbf{q}_k is, in general, a linear combination of terms of the type⁶ $\frac{\nabla_l \Psi(\mathbf{q}; t)}{\Psi(\mathbf{q}; t)} \times \frac{\nabla_j \Psi^*(\mathbf{q}; t)}{\Psi^*(\mathbf{q}; t)}$ involving also particles l and j (in which l or j might coincide with k) which could be non-interacting and non-entangled with the k -th particle.

Concluding, the fundamental request that one can deal with isolated systems (a possibility which, in Bell’s words [1], has marked the birth of experimental science) is not met by the model based on Eq. (7.7), so that it turns out to be physically unacceptable. Let us analyse whether our program can still be pursued.

One can devise various ways to circumvent the just identified difficulties. Let us begin by considering some elementary cases limiting (for the moment)

⁵Actually, once the precise choice for the relative variables is made, the theory is consistently and uniquely defined and turns out to be Galilei covariant and could be expressed in terms of the absolute coordinates and the associated derivatives, avoiding even to mention the relative variables.

⁶Again, as stated in footnote 1, we use \mathbf{q} as a shorthand for $(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$.

our considerations to models which are covariant only under the transformations of the proper group. There are two possible strategies one can follow. The first consists in building expressions for the additional velocity field associated to a given absolute coordinate which involve only derivatives with respect to the same coordinate (contrary to what happens for the previous model) and which, moreover, in the case in which the wavefunction in the absolute coordinates factorizes in two (or more) factors, depend only on the factor to which the coordinate under consideration belongs (this typically happens for expressions like $\frac{\nabla_k \Psi(\mathbf{q}; t)}{\Psi(\mathbf{q}; t)}$, k being the considered particle). Another strategy allows the appearance of derivatives referring also to other variables, provided they are entangled with the one under consideration.

To be more specific, let us list some cases. From now on, we will always make use of the language of the absolute coordinates. The general philosophy, as we know, is that of adding to the r.h.s. of Eq. (2.2) an additional velocity term as shown in Eq. (3.3). Let us consider various alternatives:

$$\mathbf{v}_{Ak}^{(1)}(\mathbf{q}; t) \propto \frac{\nabla_k |\Psi(\mathbf{q}; t)|^2}{|\Psi(\mathbf{q}; t)|^2} \times \frac{d\mathbf{v}_{Bk}(\mathbf{q}; t)}{dt} \quad (8.1)$$

$$\mathbf{v}_{Ak}^{(2)}(\mathbf{q}; t) \propto \frac{\nabla_k |\Psi(\mathbf{q}; t)|^2}{|\Psi(\mathbf{q}; t)|^2} \times \frac{d^2 \mathbf{v}_{Bk}(\mathbf{q}; t)}{dt^2} \quad (8.2)$$

$$\mathbf{v}_{Ak}^{(3)}(\mathbf{q}; t) \propto \frac{\nabla_k |\Psi(\mathbf{q}; t)|^2}{|\Psi(\mathbf{q}; t)|^2} \times \nabla_k^2 \mathbf{v}_{Bk}(\mathbf{q}; t) \quad (8.3)$$

where $\frac{d}{dt}$ is the following operator:

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \sum_k \mathbf{v}_{Bk} \cdot \nabla_k. \quad (8.4)$$

With a little algebra it is easy to check that $\nabla_k \cdot [|\Psi(\mathbf{q})|^2 \mathbf{v}_{Ak}^{(r)}] = 0$ ($r = 1, 2, 3$), so that Eq. (3.2) holds. The introduction of the time derivatives in Eqs. (8.1) and (8.2) as well as the one of the Laplace operator in Eq. (8.3) serves the purpose of guaranteeing that \mathbf{v}_{Ak} transforms correctly under Galilei boosts.

To exhibit a typical example of the second approach mentioned above, we begin by remarking that one can easily build real, positive, bounded by 1 and continuous functions w_{kj} with the property of vanishing whenever particle j is not entangled with particle k and which involve only the coordinate of those

particles which are entangled with the k -th particle. Typically an expression of the type

$$w_{kj}(\mathbf{q}; t) = 1 - \exp \left\{ -\gamma^2 [\nabla_k \cdot \nabla_j \log |\Psi(\mathbf{q}; t)|]^2 \right\} \quad (8.5)$$

(where $\gamma := \frac{\hbar}{c \sum_k m_k}$) has the desired features. One could then choose, e.g., for the additional velocity field of the k -th particle the following expression:

$$\mathbf{v}_{Ak}^{(4)}(\mathbf{q}; t) \propto \frac{\nabla_k \times \left\{ |\Psi|^2 \left[\mathbf{v}_{Bk} - \sum_j \left(\frac{w_{kj}}{\sum_r w_{kr}} \right) \mathbf{v}_{Bj} \right] \right\}}{|\Psi|^2}. \quad (8.6)$$

As it should be obvious, the coefficients in the round brackets serve the purpose of making $\mathbf{v}_{Ak}^{(4)}$ independent of the coordinates of the particles which are not entangled with the k -th one and the difference of bohmian velocities guarantees the correct transformation properties under Galilei boosts.

The just considered additional velocities have, however, wrong transformation properties for the improper transformations: $\mathbf{v}_{Ak}^{(1)}$ changes sign for space reflections and time reversal, the remaining ones do not transform correctly only for space reflection. One could circumvent this drawback by a technique analogous to the one used in Section 7, that is by multiplying γ_k times a pseudoscalar factor

$$g_k^\Psi(t) \propto \int d\mathbf{q} |\Psi(\mathbf{q}; t)|^2 \left\{ \left[\frac{\nabla_k |\Psi(\mathbf{q}; t)|}{|\Psi(\mathbf{q}; t)|} \times \nabla_k^2 \mathbf{v}_{Bk}(\mathbf{q}; t) \right] \cdot \frac{d^2 \mathbf{v}_{Bk}(\mathbf{q}; t)}{dt^2} \right\}. \quad (8.7)$$

At this point it seems that we have reached our goal. However the presence of factors of the kind of $g_k^\Psi(t)$ has some peculiar consequences we are going to discuss.

9 The implications of integrated expressions

The appearance of the factor $g_k^\Psi(t)$ in the expression for the additional velocity has some peculiar consequences arising from its involving an integral of the wavefunction over the whole space. Such consequences deserve a detailed analysis. We will, for simplicity's sake, confine our considerations to the case of a single particle (even though we will obviously have in mind a macroscopic object—typically a pointer—so that the variable \mathbf{Q} in the following formulae can be identified with the centre-of-mass coordinate of such an object).

Now we can discuss the peculiar aspects of the model of the previous Section. To this purpose let us first consider the situation with reference to the standard version of Bohmian Mechanics. There are two aspects of such a theory which deserve a detailed analysis. The first has to do with the role of the “empty waves”, the second with the problem of the identification of the *effective wave function* of a physical system. Concerning the first point, suppose that the evolved of the initial wavefunction is the sum of two terms $\Psi(\mathbf{q}; t) = \Psi_L(\mathbf{q}; t) + \Psi_R(\mathbf{q}; t)$ having, for the time interval $[t_0, T]$, disjoint supports ($\Psi_L(\mathbf{q}; t) \cdot \Psi_R(\mathbf{q}; t) = 0$). Suppose moreover that the individual physical system we are interested in *is*, at time t_0 , at a position belonging to the support of $\Psi_L(\mathbf{q}; t)$. The guidance condition for such system in the considered time interval will be determined, according to the theory, by the exact wavefunction $\Psi(\mathbf{q}; t) = \Psi_L(\mathbf{q}; t) + \Psi_R(\mathbf{q}; t)$ and by the exact differential equation

$$\frac{d\mathbf{Q}(t)}{dt} = \mathbf{v}_B(\mathbf{Q}; t) = \frac{\hbar}{m} \Im \frac{\nabla \Psi(\mathbf{q}; t)}{\Psi(\mathbf{q}; t)} \Big|_{\mathbf{q}=\mathbf{Q}}, \quad (9.1)$$

$\mathbf{v}_B(\mathbf{Q}; t)$ being the Bohmian velocity related to $\Psi(\mathbf{Q}; t)$. Now, for the specific formal structure of Bohmian Mechanics it happens that in the case under consideration we can replace $\Psi(\mathbf{Q}; t)$ by $\Psi_L(\mathbf{Q}; t)$ everywhere in the dynamical equations (see for example [3]). Obviously, no wavefunction can have compact support for a finite time interval, but the above conclusion also holds, with an accuracy which is higher the smaller is the overlapping of the two states, when the two terms $\Psi_L(\mathbf{Q}; t)$ and $\Psi_R(\mathbf{Q}; t)$ of the superposition have almost disjoint supports for the time interval $[t_0, T]$ we are interested in. In the specific case of two pointer states corresponding to two macroscopically different locations this holds for extremely long times.

We stress that the argument we have just developed has the following conceptual structure: the correct dynamics of the pointer is governed by $\Psi(\mathbf{Q}; t)$ and by Eq. (9.1); however, the fact that to an extremely high degree of accuracy one can consider the evolution as governed by $\Psi_L(\mathbf{Q}; t)$ and by the corresponding Bohmian velocity field $\mathbf{v}_{BL}(\mathbf{Q}; t) = \frac{\hbar}{m} \Im \frac{\nabla \Psi_L(\mathbf{Q}; t)}{\Psi_L(\mathbf{Q}; t)}$ makes immediately evident that the theory solves the macro-objectification problem, in particular that macroscopic objects like pointers in a superposition of spatially separated states evolve as if the “empty wave” referring to the region not containing the pointer would not exist⁷.

⁷Anyways it has to be kept in mind that the exact dynamics is determined by $\Psi(\mathbf{Q}; t)$ and

Let us analyse now the second relevant feature of Bohmian Mechanics, i.e. the way in which it deals with the problem of the identification of the *effective wavefunction* [5] of our system. In synthesis the question can be summarized as follows: the system under consideration (as well as an assembly of such systems) is *de facto* a subsystem of the (unique) universe in which we live. In general the wavefunction of the universe at time t_0 will have a form implying an entanglement of the system we are considering and other parts of the universe itself:

$$\tilde{\Psi}(\mathbf{Q}, \mathbf{q}_j; t_0) = \sum_s \Psi_s(\mathbf{Q}; t_0) \Phi_s(\mathbf{q}_j; t_0), \quad (9.2)$$

where the wavefunctions $\Psi_s(\mathbf{Q}; t_0)$ correspond to the macroscopically different situations of the pointer and the coordinates \mathbf{q}_j refer to the particles of the environment of the system under consideration.

An essential step is that of proving the legitimacy of associating to an individual system — the pointer with coordinate \mathbf{Q} (or an ensemble of such systems) — a specific one of the wavefunctions $\Psi_s(\mathbf{Q}; t_0)$ of Eq. (9.1). The precise sense in which this is appropriate has been discussed in great detail in a series of beautiful papers [5] by Dürr et al. The argument requires two steps: first of all, following a line of thought analogous to the one described above in connection with the problem of the empty wave, one proves that it is legitimate to keep only the term of Eq. (9.2) corresponding to the actual location of the pointer. Secondly, one recalls that in the case of factorized statevectors the guidance equations for the coordinates of one of the two factors do not involve the other factor. Thus, one can actually consider a specific $\Psi_s(\mathbf{Q}; t_0)$ as the effective initial wavefunction of the system we are interested in.

We can now compare the just discussed feature of the standard theory with those of the alternative versions of it which are the subject of the present paper. When expressions like $g_k^\Psi(t)$ appear, the situation turns out to be quite different. In fact, with reference to the first of the points discussed above, the exact guidance equation is:

$$\frac{d\mathbf{Q}(t)}{dt} = \mathbf{v}_B(\mathbf{Q}; t) + \mathbf{v}_A(\mathbf{Q}; t) \quad (9.3)$$

by Eq. (9.1); as it is obvious, if one considers times $t \gg T$ so large that the spreading of the wave packets leads to an appreciable overlapping, the replacement of $\Psi(\mathbf{Q}; t)$ by $\Psi_L(\mathbf{Q}; t)$ would be no more legitimate.

showing that the intensity of the velocity field for a particle in the support of $\Psi_L(\mathbf{Q}; t)$ depends crucially also on the empty wave $\Psi_R(\mathbf{Q}; t)$.

The conclusion is that the appearance of integrated expressions destroys two of the most important features of Bohmian Mechanics, i.e. the possibility of disregarding the empty waves and the one of resorting to the effective wavefunction⁸.

Concluding, the appearance of integrated expressions has physically unacceptable implications. In the next Section we will show how to get rid of this bad feature of the just considered model.

10 A more palatable model

As everybody knows, the curl of a pseudovector transforms like a vector; this suggests to consider, e.g., the following choice for the additional velocity field, which is directly obtained from the expression⁹ (8.3):

$$\mathbf{v}_{Ak}(\mathbf{q}; t) \propto \frac{\nabla_k \times (\nabla_k |\Psi(\mathbf{q}; t)|^2 \times \nabla_k^2 \mathbf{v}_{Bk}(\mathbf{q}; t))}{|\Psi(\mathbf{q}; t)|^2}. \quad (10.1)$$

Analogously, one could use the same trick on the expression (8.6):

$$\mathbf{v}_{Ak}(\mathbf{q}; t) \propto \frac{\nabla_k \times \left\{ \nabla_k \times \left\{ |\Psi|^2 \left[\mathbf{v}_{Bk} - \sum_j \left(\frac{w_{kj}}{\sum_r w_{kr}} \right) \mathbf{v}_{Bj} \right] \right\} \right\}}{|\Psi|^2}. \quad (10.2)$$

The two just considered forms (10.1) and (10.2) for the additional velocity field do not exhibit any of the unacceptable features we have analysed in the previous Sections. More specifically, they have the correct transformation properties under the full Galilei group, they guarantee that the dynamics of a group of particles which are non-interacting and not entangled with other particles is independent of the last, and they consent to disregard the empty

⁸With reference to this second feature we note that the appearance of integrated expressions makes illegitimate the replacement of the superposition (9.2) with one of its terms, it does not alter the fact that in the case of a factorized wavefunction one can disregard one of the two factors to describe the dynamics of the coordinates of the other factor, since if $\Psi = \psi(q_k) \cdot \phi(\dots)$, one has $g_k^\Psi(t) = g_k^\psi(t)$.

⁹A completely analogous trick can obviously be used with reference to the expression (8.1).

wave in the case of far away superpositions of states associated to a macroscopic object. Obviously, what we have presented are only two of many other possible choices we will not consider since our purpose is not to identify the most general bohmian-like model, but only to make clear that different alternatives are possible.

11 The case of particles with spin

If one limits one's considerations to particles with spin, then the elaboration of alternative models to Bohmian Mechanics becomes even easier. Actually Bohm and Hiley themselves [4] entertained the idea of theories exhibiting trajectories differing from the standard bohmian ones but reproducing the quantum configuration density distributions in the case of a particle of spin 1/2. The procedure is quite simple, it consists in adding to the velocity field:

$$\mathbf{v}_B(\mathbf{q}; t) = \frac{\hbar}{2mi} \frac{\Psi^\dagger(\mathbf{q}; t) \nabla \Psi(\mathbf{q}; t) - [\nabla \Psi^\dagger(\mathbf{q}; t)] \Psi(\mathbf{q}; t)}{\Psi^\dagger(\mathbf{q}; t) \Psi(\mathbf{q}; t)}, \quad (11.1)$$

proposed by Bohm and by Bell [10] as a straightforward generalisation of the bohmian prescription, an additional velocity field having the following form:

$$\mathbf{v}_A(\mathbf{q}; t) \propto \frac{\nabla \times (\Psi^\dagger(\mathbf{q}; t) \boldsymbol{\sigma} \Psi(\mathbf{q}; t))}{\Psi^\dagger(\mathbf{q}; t) \Psi(\mathbf{q}; t)}. \quad (11.2)$$

The resulting velocity field

$$\mathbf{v}_N(\mathbf{q}; t) = \mathbf{v}_B(\mathbf{q}; t) + \mathbf{v}_A(\mathbf{q}; t) \quad (11.3)$$

is easily proved to have all properties which guarantee the continuity and regularity of the trajectories, as well as all desired properties to ensure the necessary covariance properties of the resulting theory. The choices (11.2) and (11.3) represent only one of many others which one can easily make by taking advantage, e.g., of some of the proposals we have presented in the previous Sections.

Just to exhibit explicit examples, let us consider all particles with spin 1/2 of our system and the associated velocity fields $\mathbf{v}_{Bj}(\mathbf{q}; t)$ of Standard Bohmian Mechanics:

$$\mathbf{v}_{Bj}(\mathbf{q}; t) = \frac{\hbar}{2mi} \frac{\Psi^\dagger(\mathbf{q}; t) \nabla_j \Psi(\mathbf{q}; t) - [\nabla_j \Psi^\dagger(\mathbf{q}; t)] \Psi(\mathbf{q}; t)}{\Psi^\dagger(\mathbf{q}; t) \Psi(\mathbf{q}; t)}. \quad (11.4)$$

In terms of these velocities we define scalar quantities w_{kj} (k and j running over all particles of spin 1/2) according to Eq. (8.5). Such quantities vanish whenever the considered particles are not entangled with each other. For all spin 1/2 fermions of our system we add to the velocity field $\mathbf{v}_{Bj}(\mathbf{q}; t)$ an additional velocity field $\mathbf{v}_{Ak}(\mathbf{q}; t)$ according to:

$$\mathbf{v}_{Ak}(\mathbf{q}; t) \propto \frac{\nabla_k \times (\Psi^\dagger(\mathbf{q}; t) \sum_j w_{kj} \boldsymbol{\sigma}^{(k)} \times \boldsymbol{\sigma}^{(j)} \Psi(\mathbf{q}; t))}{\Psi^\dagger(\mathbf{q}; t) \Psi(\mathbf{q}; t)}. \quad (11.5)$$

In the above equation $\boldsymbol{\sigma}^{(k)}$ and $\boldsymbol{\sigma}^{(j)}$ are the vector spin operators of particles k and j respectively.

As one can immediately see, the additional velocities satisfy all necessary requirements concerning their transformation properties for the extended Galilei group. The additional velocity for a fermion depends only on the positions of the fermions with which it is entangled. Furthermore the model solves the macro-objectification problem just as Bohmian Mechanics does.

The conclusion should be obvious: the model we have just introduced has exactly all the same features of the corresponding Standard Bohmian model implying however different trajectories for all particles of spin 1/2. It goes without saying that the same procedure can be used to modify the trajectories of all particle having spin since the transformation properties of the additional velocity do not depend on the value of the spin.

12 The problem of the nodes of the wavefunction

The models introduced in Sections 10 and 11 are acceptable candidates for alternative bohmian-like theories equivalent to Quantum Mechanics. The only feature which could give rise to problems derives from the fact that since we know [6] that the bohmian velocity field can exhibit divergences in the set \mathcal{N} of the nodes of Ψ and since the alternative models contain derivatives of such fields, they could give rise to physically unacceptable consequences. We do not intend to analyse this point in detail. However we remark that under the conditions imposed to the potential and to the initial wavefunction in [6] the authors have been able to prove that the bohmian velocity field $\mathbf{v}_{Bk}(\mathbf{q}; t)$ at any time is of class C^∞ on the complement of \mathcal{N} . This in turn implies

that the functions $\nabla_k^2 \mathbf{v}_{Bk}(\mathbf{q}; t)$ and w_{kj} appearing in (10.1), (10.2) and (11.5) can be singular only on \mathcal{N} . Since the function¹⁰ $\exp \left[- \left(\frac{\hbar}{m_k c} \frac{\nabla_k |\Psi|^2}{|\Psi|^2} \right)^2 \right]$ vanishes exponentially at the nodes of Ψ , if one inserts such a function as a factor in the argument of the curl appearing in the above equations, one gets rid of any singular behaviour of the additional velocity on the set \mathcal{N} . Concluding, we have shown that there actually exist perfectly acceptable (even though formally less simple and elegant) alternatives to Bohmian Mechanics.

13 The guidance view versus the Newtonian view of Bohmian Mechanics

In this Section we take into account some recent remarks about the interpretation of Bohmian Mechanics, which have been presented by Baublitz and Shimony [11] and which have been reiterated by Shimony himself in private correspondence on the content of the present paper. The position of these authors can be summarized in the following terms:

- There are two possible attitudes towards Bohmian Mechanics, the Newtonian one, in which one considers a classical equation of motion involving the acceleration of the particle(s) and adds the quantum potential to the classical one. According to the authors, the main advantage of this approach derives from its being more classical. However — they also point out — within a Newtonian picture one should consider the initial velocities of the particles as contingencies, so that the request that they satisfy the initial guidance condition

$$\mathbf{p}(0) = \nabla S(\mathbf{r}; 0) \tag{13.1}$$

(S/\hbar being the initial phase of the wavefunction) does not fit within such a picture. In spite of this, the authors of [11] seem to prefer this view

¹⁰Note that this function is positive and bounded by 1, is invariant under the extended Galilei group and has the good property underlined in the previous Sections concerning the factorizability of the wavefunction. Actually, to be rigorous, one should take into account that, as S. Goldstein suggested us, in very pathological cases as for example a wavefunction having an accumulation point X of nodes, the factor we propose to insert could have no limit for its argument tending to X. However, when similar situations are taken into account, problems can arise also in Bohmian Mechanics.

with respect to the position which considers only the general guidance equation we have presented in Section 1 and which is adopted by modern supporters of Bohmian Mechanics. Baublitz and Shimony hope, with Bohm, that one could get rid of the condition (13.1) by adding nonlinear terms to the evolution equation which should lead to the satisfaction of the guidance condition in a short time, independently of the initial choice for the velocities. In private correspondence Shimony has raised the question of whether a similar program could be developed for the *alternative* bohmian models we are considering here.

- The same authors analyse the guidance version of Bohmian Mechanics and they recognise that, in a sense, it is more consistent than the Newtonian one. However — they stress — such an approach is less palatable because, if classically interpreted, it seems to imply that the force on a system determines its velocity and not its acceleration. From a philosophical standpoint they see the guidance view as more Aristotelian and as such less satisfactory than the Newtonian interpretation.

Our personal opinion on this delicate matter is quite different. Actually we consider as misleading the attempts to present Bohmian Mechanics as an *almost classical* theory. We agree perfectly with Bell [12] that it is a scandal that Bohmian Mechanics is not presented in courses of Quantum Mechanics and precisely for the reason that its consideration helps the student to clearly grasp the dramatic divergences between the classical and the quantum views of natural phenomena. Why should one try to maintain that, after all, Bohmian Mechanics represents some sort of integration of quantum phenomena in a classical picture when we all know very well that its most fundamental and essential features are nonclassical? Does not such a suggestion mislead the student making it more difficult for him to fully appreciate the peculiar nonclassical aspects of nonlocality, contextuality and of the unavoidable invasivity of any measurement, which characterise the bohmian picture of physical processes?

These statements should have made clear why we adhere to the guidance interpretation of the formalism. We would like to add some comments. The proposal of a modified theory in which the guidance condition is not imposed but emerges as a consequence of some specific dynamical feature can be usefully contemplated but, at the present stage, it represents more wishful thinking than a serious scientific attempt. Furthermore, the problem we want to tackle

in this paper should be very clear: are there *alternative* theories to Bohmian Mechanics attributing, as it does, *deterministic* trajectories to the particles and which turn out to be equivalent to SQM? We have deliberately chosen not to consider the possibility of theories which, in one way or another, contradict quantum predictions. In fact, in our opinion, if one chooses to contemplate such a possibility it is more interesting to follow the line of thought of the dynamical reduction program [13] than the one of the incompleteness of the Hilbert space description of individual physical systems.

Having stated that, we recognise that the questions raised by Shimony of whether one can consider the Newtonian version of the alternative models we are envisaging and whether, within such a version, one could consider nonlinear modifications leading to the dynamical satisfaction of the guidance condition are surely interesting. Concerning the first point the answer is clearly affirmative, as everybody would easily guess. We have presented in Appendix B the Hamilton–Jacobi version of our alternative model. From this point of view one can also consider the whole family of equivalent bohmian-like theories with different trajectories as a family of Newtonian models of Quantum Mechanics. Concerning the second point we call attention to the fact that, at the moment, there are no indications that it can be consistently followed even for the standard bohmian version of the theory; therefore we consider extraneous to the purpose of this paper to tackle this problem. It could be an interesting task for people committed to the Newtonian interpretation of the model. However, we cannot avoid stressing that we do not see any reason whatsoever why, in case the indicated line would turn out to be viable, it would be so for Bohmian Mechanics and not for its modifications we have considered here.

14 The Conceptual Implications of the Previous Results

In this Section we will analyse the implications of the fact that there exist infinitely many deterministic theories (even if they can look much more exotic and/or less elegant) which account for physical processes in terms of particles which follow precise trajectories, all of which reproduce the configuration density distributions of SQM, satisfy all possible physically meaningful requirements one can put forward for them and nevertheless attribute different

trajectories to the particles.

To put the attempts we have presented in this paper in the appropriate light, it is useful to make precise the general perspective which we adopt. We are interested in theories which attribute a privileged role to the positions of the particles. Thus, the common feature of all the theories we will analyse will be that of yielding a configuration density distribution coinciding, at any time, with the one of SQM, i.e. $|\Psi(\mathbf{q}; t)|^2$, \mathbf{q} being as usual a shorthand for the position variables of all the particles of the universe.

Obviously, within the considered class of possible theoretical models, one should make precise in which sense they must yield the desired result. From this point of view we can consider three different general approaches:

1. Deterministic models, i.e. models à la Bohm, in which the particles follow precise trajectories uniquely determined by the initial positions. The stochastic features of such models derive simply from the lack of knowledge that one has about the precise initial positions of the particles.
2. Stochastic models [17,18] which are based on a simple idea, i.e. that particles have positions and follow trajectories, but contain irreducible (i.e. non-epistemic) random elements. Typically, within such an approach, the knowledge of the wavefunction and the initial position of a particle allows only to make probabilistic assertions about its future position.
3. The extreme case we can consider is the one in which one claims that all particles have definite positions at all times but he does not commit himself, or even he denies the possibility of considering trajectories, i.e. of connecting the present position of a particle to its past position. The possibility of taking such a point of view has been stressed by Bell [19] in connection with a particular attitude about the Many World Interpretation of SQM, according to which *one can redistribute the configuration \mathbf{q} at random (with weight $|\Psi(\mathbf{q}; t)|^2$) from one instant to the next*. Such a position can be consistently taken for, as stated by Bell himself, *we have not access to the past, but only to memories, and these memories are just part of the instantaneous configuration of the world*.

With reference to the three positions just outlined we stress the following points. If one takes position 3 one simply denies the possibility of describing the world in terms of particle trajectories. A detailed discussion of position 2

has been given in ref. [20]. For what concerns the problem we are analysing in this paper we note that Goldstein, Dürr and Zanghì have proved [21] that in Stochastic Mechanics one can exhibit infinitely many equivalent theories (again in the sense that they give rise to the same distribution $|\Psi(\mathbf{q}; t)|^2$) which are characterised by the fact that the families of possible trajectories differ. To give an idea of the procedure we recall that Nelson's approach identifies the quantum evolution with a classical diffusion process of the Wiener type characterised by a drift $\mu(\mathbf{q}; t)$ (which in turn is determined by the initial wavefunction $\Psi(\mathbf{q}; t_0)$) and variances σ_k^2 related to the Planck constant and to the mass of the various particles according to $\sigma_k^2 = \frac{\hbar}{m_k}$. The above mentioned authors have proved that one can change the variance and the drift, getting however the same density distribution. In particular, in the limit $\sigma_k^2 \rightarrow 0$ the resulting theory has deterministic trajectories and actually coincides with Bohmian Mechanics; in the limit $\sigma_k^2 \rightarrow \infty$ the trajectories lose any direct physical significance so that one is left only with the density distribution $|\Psi(\mathbf{q}; t)|^2$.

The analysis of the present paper shows that a similar situation occurs in the case of theories with deterministic trajectories, i.e. that Bohmian Mechanics is a member of an equivalence class of theories, all reproducing the desired density distribution $|\Psi(\mathbf{q}; t)|^2$. Physically motivated conditions, such as the genuine covariance requirement and the others we have analysed in this paper, restrict remarkably the class of deterministic models of the considered type but they do not allow to pick up one member of the equivalence class as physically preferable to all the others (obviously one can say that Bohmian Mechanics is more simple and more elegant but we do not want to rely on such criteria). This being the situation, we think it appropriate to conclude with some general remarks.

We are not at all worried by the fact that one can have empirically equivalent but essentially different theories. Actually this is precisely what happens with Bohmian Mechanics and Quantum Mechanics. An analogous situation occurs in relation with the dynamical reduction models. In fact, one can certainly build different theories which are all consistent with all known facts about the behaviour of microsystems and which lead to the macro-objectification of macroscopic properties in quite similar ways. The best explicit examples are the discrete [13] (GRW) and continuous [15] (CSL) versions of spontaneous localisation models or the original CSL model [15] and those relating reduction to gravity or mass density [16].

In spite of these facts we cannot avoid calling attention on some relevant differences between the just mentioned cases (and many other similar ones which could be mentioned) and the one we have outlined in this paper. For instance, concerning the relations between Quantum Mechanics and Bohmian Mechanics, even though it is true that they turn out to be empirically equivalent, the ontologies behind them are extremely different, such differences representing the main point of interest of Bohmian Mechanics since they lead to two completely different views about natural phenomena. In particular, Bohmian Mechanics allows one to consider as fundamentally epistemic the quantum probabilities, while SQM insists on their nonepistemic nature. Bohmian Mechanics allows a unified description of all natural processes while SQM suffers of an unavoidable ambiguity about the facts of our experience. Coming to the dynamical reduction models these qualify themselves as rival theories to SQM rather than reinterpretations of it, and can, in principle, be experimentally tested against Quantum Mechanics itself. Moreover the above mentioned many variants of the dynamical reduction theories have, actually, different physical implications, and, in any case, embody physically different mechanisms (in spite of the fact that it can turn out to be extremely difficult to discriminate among them).

The situation is quite different for the case we have considered in this paper. We are confronted with alternative theories which are empirically equivalent and have exactly the same ontology: to use Bell's words what these theories are about, what they speak of, is the fundamental fact that particles *have definite positions at all times and move along precise trajectories* (which, however, as we all know, cannot be detected).

We can then imagine various possible scenarios:

- (a) Our hope is that someone (more expert than we are on this topic) could find physically meaningful and compelling criteria to identify, among the class of theories *equivalent* to Bohmian Mechanics in the sense we have made precise, one which is physically privileged. One could, e.g., take the argument of Bohm and Hiley as implying that the nonrelativistic limit of Dirac equation seems to favour a non-standard bohmian trajectory with respect to the standard one. Such a remark could be used to identify criteria to pick up the best model among all possible ones. If this would be feasible, then one would have surely achieved a deeper understanding of the theory.

- (b) Alternatively, it could happen that, at the nonrelativistic level characterising the present analysis, no compelling criterion (besides formal simplicity) exists making one of the equivalent versions physically better grounded than the others. In this case one would be led to the conclusion that, actually, the appropriate bohmian-like reformulation of Quantum Mechanics is given by the equivalence class $[\mathcal{B}]$ of all theories which attribute definite (but different) trajectories to the particles, reproducing the probability density distribution of SQM. This fact would not be distressing and could turn out to be only temporary and of some use. In fact, it could very well happen that the hypothetical relativistic generalisations of the theory would not be affected by the above ambiguity. One could even hope that the greater variety of nonrelativistic bohmian-like theories discussed in this paper could help in finding a satisfactory relativistic generalisation. Having more freedom in the theory one is dealing with usually is an advantage and may allow to circumvent more easily some difficulties.
- (c) For completeness we cannot avoid mentioning that one, taking a radical position about our analysis, could consider the fact that many inequivalent theories with different trajectories can be exhibited as an indication that one must not consider the trajectories since they are ill defined. In other words one could be led to take a position like the one we have mentioned above as position 3.

We do not share such an attitude and we stress that, in the last issue, it would correspond to taking a quite trivial and ambiguous perspective about the conceptual problems of Quantum Mechanics. The best illustration of this fact is given once more by a sentence by Bell [19]: *Everett's replacement of the past by memories is a radical solipsism — extending to the temporal dimension the replacement of everything outside my head by my impressions, of ordinary solipsism or positivism. Solipsism cannot be refuted. But if such a theory were taken seriously it would hardly be possible to take anything else seriously. ... It is always interesting to find that solipsists and positivists, when they have children, have life insurance.* This shows that the possibility just envisaged would turn out to be almost devoid of any physical significance and would practically amount to an *a priori* refusal of the bohmian program.

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Appendix

A Gauge Transformations

Suppose we consider the Hamiltonian $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{r}})$ and let us take into account the solution $|\Psi; t\rangle$ of the Schrödinger equation:

$$i\hbar \frac{\partial |\Psi, t\rangle}{\partial t} = \hat{H} |\Psi; t\rangle; \quad (\text{A.1})$$

then we know that for any real function $\alpha(\hat{\mathbf{r}})$:

$$|\tilde{\Psi}; t\rangle = e^{i\alpha(\hat{\mathbf{r}})} |\Psi; t\rangle \quad (\text{A.2})$$

is a solution of the Schrödinger equation for the Hamiltonian:

$$\tilde{H} = e^{i\alpha(\hat{\mathbf{r}})} \hat{H} e^{-i\alpha(\hat{\mathbf{r}})} = \frac{[\hat{\mathbf{p}} - \hbar \nabla \alpha(\hat{\mathbf{r}})]^2}{2m} + V(\hat{\mathbf{r}}), \quad (\text{A.3})$$

i.e.:

$$i\hbar \frac{\partial |\tilde{\Psi}, t\rangle}{\partial t} = \tilde{H} |\tilde{\Psi}; t\rangle. \quad (\text{A.4})$$

Note that, since the coordinate representations of $|\Psi, t\rangle$ and $|\tilde{\Psi}, t\rangle$ are $\Psi(\mathbf{r}, t)$ and $\tilde{\Psi}(\mathbf{r}, t) = e^{i\alpha(\hat{\mathbf{r}})} \Psi(\mathbf{r}, t)$ respectively, one has $|\tilde{\Psi}(\mathbf{r}, t)|^2 = |\Psi(\mathbf{r}, t)|^2 \forall (\mathbf{r}, t)$, and the configuration density distribution is then the same at all times and in all places for the two physical systems under examination.

Let us now evaluate the velocity field associated to $\tilde{\Psi}(\mathbf{r}; t)$ by using the standard expression of Bohmian Mechanics associated to Eq. (A.1):

$$\tilde{\mathbf{v}}_B(\mathbf{r}; t) = \frac{\hbar}{m} \Im \frac{\tilde{\Psi}^*(\mathbf{r}; t) \nabla \tilde{\Psi}(\mathbf{r}, t)}{|\tilde{\Psi}(\mathbf{r}; t)|^2}; \quad (\text{A.5})$$

we then have:

$$\tilde{\mathbf{v}}_B(\mathbf{r}; t) = \frac{\hbar}{m} \Im \frac{\Psi^*(\mathbf{r}, t) \nabla \Psi(\mathbf{r}, t)}{|\Psi(\mathbf{r}, t)|^2} + \frac{\hbar}{m} \nabla \alpha(\mathbf{r}). \quad (\text{A.6})$$

However, it has to be mentioned that when the Hamiltonian contains momentum terms of the type we have introduced, the conserved current density for the Schrödinger equation has to be changed (as one can easily check) according to:

$$\mathbf{j}(\mathbf{r}; t) \longrightarrow \mathbf{j}(\mathbf{r}; t) - \frac{\hbar}{m} |\Psi(\mathbf{r}, t)|^2 \nabla \alpha(\mathbf{r}). \quad (\text{A.7})$$

Since the velocity is given by the ratio of the current and the density, we see then that the appropriate velocity field for the wavefunction $\tilde{\Psi}(\mathbf{r}, t)$ is:

$$\begin{aligned} \tilde{\mathbf{v}}_B(\mathbf{r}, t) &= \frac{\hbar}{m} \Im \frac{\tilde{\Psi}^*(\mathbf{r}, t) \nabla \tilde{\Psi}(\mathbf{r}, t)}{|\tilde{\Psi}(\mathbf{r}, t)|^2} - \frac{\hbar}{m} \nabla \alpha(\mathbf{r}) \\ &= \frac{\hbar}{m} \Im \frac{\Psi^*(\mathbf{r}, t) \nabla \Psi(\mathbf{r}, t)}{|\Psi(\mathbf{r}, t)|^2} + \frac{\hbar}{m} \nabla \alpha(\mathbf{r}) - \frac{\hbar}{m} \nabla \alpha(\mathbf{r}) \\ &= \frac{\hbar}{m} \Im \frac{\Psi^*(\mathbf{r}, t) \nabla \Psi(\mathbf{r}, t)}{|\Psi(\mathbf{r}, t)|^2} = \mathbf{v}_B(\mathbf{r}, t), \end{aligned} \quad (\text{A.8})$$

as it had to be, since the transformation we have considered corresponds simply to changing the phases of the positions eigenstates, leaving the physics unchanged.

Thus, even though a gauge transformation implies a change in the wavefunction (but not in the associated probability density), within a bohmian picture it does not change the trajectories of the particles, an important fact for the alternatives to the bohmian theory we are envisaging in this paper.

B Hamiltonian Formulation of the Theory

In this appendix we will show how it is possible to give a hamiltonian formulation to our alternative theory, following the same procedure pursued by Bohm [2] in 1952.

First of all let us consider for simplicity a system of one particle and take the Bohm point of view writing Ψ as

$$\Psi(\mathbf{r}; t) = R(\mathbf{r}; t) \exp \left[\frac{i}{\hbar} S(\mathbf{r}; t) \right]. \quad (\text{B.1})$$

Then the Schrödinger equation is equivalent to:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + Q = 0 \quad (\text{B.2})$$

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right) = 0 \quad (\text{B.3})$$

where V and $Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$ are the actual and the quantum potentials, respectively.

Since the bohmian momentum is given by:

$$\mathbf{p} = m\mathbf{v} = \nabla S \quad (\text{B.4})$$

we obtain that the motion implied by Eq. (B.4) is equal to that governed by the Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V + Q$.

Taking the same point of view, we note that Eq. (B.1) can also be rewritten as:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S + m\mathbf{v}_A)^2}{2m} + V + \tilde{Q} = 0 \quad (\text{B.5})$$

with $\tilde{Q} = Q - \frac{m}{2}\mathbf{v}_A^2 - \nabla S \cdot \mathbf{v}_A$. It is then clear that Eq. (B.5) can be considered as the Hamilton-Jacobi equation for a particle moving in a velocity dependent generalised potential, like for example a particle of charge $-e$ in a magnetic field with a vector potential $\mathbf{A} = -\frac{mc}{e}\mathbf{v}_A$ and a scalar potential $\tilde{V} = V + \tilde{Q}$. Therefore we have shown that also our trajectories admit a Hamiltonian description. We conclude by proving that the generalised potential relative to the system described by Eq. (B.5) is given by

$$\tilde{V}_{\text{gen}} = \tilde{V} + m\mathbf{v} \cdot \mathbf{v}_A. \quad (\text{B.6})$$

In fact, taking into account Eq. (B.5) (more precisely the equation obtained by taking the gradient of the two sides of Eq. (B.5)), it is simple to check that:

$$\begin{aligned} m \frac{dv_{Nx}}{dt} &= -\frac{\partial \tilde{V}_{\text{gen}}}{\partial x} + \frac{d}{dt} \frac{\partial \tilde{V}_{\text{gen}}}{\partial v_x} \\ &= -\frac{\partial \tilde{V}}{\partial x} - m\mathbf{v} \cdot \frac{\partial \mathbf{v}_A}{\partial x} + m \frac{dv_{Ax}}{dt} \end{aligned}$$

$$= \frac{\partial}{\partial t}(\partial_x S) + \frac{\nabla S}{m} \cdot \nabla(\partial_x S) + \mathbf{v}_A \cdot \partial_x(\nabla S) + m \frac{dv_{Ax}}{dt} \quad (\text{B.7})$$

$$= \frac{\partial}{\partial t}(\partial_x S) + \frac{\nabla S + m\mathbf{v}_A}{m} \cdot \nabla(\partial_x S) + m \frac{dv_{Ax}}{dt}$$

$$= \frac{d}{dt}(\partial_x S) + m \frac{dv_{Ax}}{dt}$$

$$= m \frac{d}{dt}(v_{Bx} + v_{Ax}). \quad (\text{B.8})$$

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